

EXISTENCE OF COMMON AND UPPER FREQUENTLY HYPERCYCLIC SUBSPACES

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ABSTRACT. We provide criteria for the existence of upper frequently hypercyclic subspaces and for common hypercyclic subspaces, which include the following consequences. There exist frequently hypercyclic operators with upper-frequently hypercyclic subspaces and no frequently hypercyclic subspace. On the space of entire functions, each differentiation operator induced by a non-constant polynomial supports an upper frequently hypercyclic subspace, and the family of its non-zero scalar multiples has a common hypercyclic subspace. A question of Costakis and Sambarino on the existence of a common hypercyclic subspace for a certain uncountable family of weighted shift operators is also answered.

1. INTRODUCTION

Throughout this paper, X and Y denote separable infinite-dimensional Fréchet spaces, and $L(X, Y)$ denotes the space of continuous linear operators from X to Y . When $Y = X$, we let $L(X) = L(X, X)$. A sequence (T_n) in $L(X, Y)$ is said to be *hypercyclic* provided there is some x in X for which the set $\{T_n x : n \geq 0\}$ is dense in Y . Such vector x is called a *hypercyclic vector* for (T_n) . An operator T in $L(X)$ is said to be hypercyclic whenever its sequence of iterates (T^n) is a hypercyclic sequence.

An important question about a hypercyclic operator is whether it supports any infinite-dimensional *closed* subspace in which every non-zero vector is hypercyclic. Such a subspace is called a *hypercyclic subspace*. This notion is interesting –see [6, Chapter 8] or [28, Chapter 10]– because some hypercyclic operators do not support hypercyclic subspaces while some other operators do. Indeed, Montes [35] showed that no scalar multiple of the backward shift on ℓ^p ($1 \leq p < \infty$) supports a hypercyclic subspace. On the other hand, the collective work of Bernal and Montes [10], Petersson [36], Shkarin [40] and Menet [32] shows that each non-scalar convolution operator on the space $H(\mathbb{C})$ of entire functions has a hypercyclic subspace. Convolution operators on $H(\mathbb{C})$ are those that commute with the operator D of complex differentiation, and are precisely those of the form

$$\phi(D) : H(\mathbb{C}) \rightarrow H(\mathbb{C}), f \mapsto \sum_{n \geq 0} a_n D^n f,$$

where $\phi(z) = \sum_{n \geq 0} a_n z^n \in H(\mathbb{C})$ is of exponential type [25].

Date: August 23rd, 2014.

2010 Mathematics Subject Classification. Primary 47A16.

Key words and phrases. Common hypercyclic vectors; Hypercyclic subspaces; Frequently hypercyclic operators.

The first author is partially supported by MEC and FEDER Projects MTM2007-64222 and MTM2010-14909. The second author is supported by a grant of FRIA.

1.1. Two sufficient criteria for the existence of hypercyclic subspaces.

Several criteria for the existence or the non-existence of hypercyclic subspaces are known; we will be particularly interested in the next two theorems. We first recall the following.

Definition 1.1. (León and Müller [31]) A sequence (T_n) in $L(X, Y)$ satisfies *Condition (C)* along a given strictly increasing sequence (n_k) of positive integers provided

- (1) For each x in some dense subset of X we have $T_{n_k}x \rightarrow 0$, and
- (2) For each continuous seminorm p on X , $\bigcup_k T_{n_k}(p^{-1}[0, 1])$ is dense in Y .

First formulated on the Banach space setting, Condition (C) ensures by a standard Baire Category argument that the sequence (T_n) has a residual set of hypercyclic vectors whenever Y is separable [32].

Theorem 1.2 (Criterion M_0 [32]). *Let (T_n) be a sequence in $L(X, Y)$ satisfying Condition (C) along a sequence (n_k) . Suppose that X supports a continuous norm and that there exists an infinite-dimensional closed subspace M_0 of X such that*

$$T_{n_k}x \rightarrow 0 \quad \text{for all } x \in M_0.$$

Then (T_n) has a hypercyclic subspace.

Definition 1.3. Let (M_n) be a sequence of infinite-dimensional closed subspaces of X with $M_n \supseteq M_{n+1}$ for all n . A sequence (T_n) in $L(X, Y)$ is said to be *equicontinuous along (M_n)* provided for each continuous seminorm q on Y there exists a continuous seminorm p of X so that for each $n \in \mathbb{N}$

$$q(T_n x) \leq p(x) \quad \text{for each } x \in M_n.$$

Theorem 1.4 (Criterion (M_k) [32]). *Let (T_n) be a sequence in $L(X, Y)$ satisfying Condition (C) along a sequence (n_k) . Suppose that X supports a continuous norm and that (T_{n_k}) is equicontinuous along some non-increasing sequence (M_k) of closed, infinite-dimensional subspaces of X . Then (T_n) has a hypercyclic subspace.*

The first versions of Criterion M_0 and Criterion (M_k) appeared under the stronger assumption of satisfying the Hypercyclicity Criterion instead of Condition (C). Criterion M_0 first appeared in 1996 for the case of operators on Banach spaces and is due to Montes [35]; this was generalized to operators on Fréchet spaces with a continuous norm by Bonet, Martínez and Peris [15] and independently by Petersson [36], as well as to sequences of operators on Banach spaces by León and Müller [31]. The present version on sequences of operators on Fréchet spaces with a continuous norm is due to Menet [32]. Criterion (M_k) first appeared implicitly for the case of operators on Hilbert and Banach spaces (and where the sequence (M_k) is constant) in the works of León and Montes [29] and by González et al [26], respectively; this was extended in 2006 by León and Müller [31] to sequences of operators on Banach spaces and more recently by Menet [32] to the present version on Fréchet spaces with a continuous norm. Both Criterion M_0 and Criterion (M_k) have generalizations to Fréchet spaces without continuous norm as well [33].

1.2. Two themes about hypercyclic subspaces we consider. This paper explores a link between Criterion M_0 and Criterion (M_k) and its consequences within two themes: subspaces of \mathcal{U} -frequently hypercyclic vectors and subspaces of common hypercyclic vectors. The first theme is motivated by recent work of Bonilla and

Grosse-Erdmann [18] that initiated the study of *frequently hypercyclic subspaces*, that is, of hypercyclic subspaces consisting entirely (but the origin) of frequently hypercyclic vectors. The notions of frequently hypercyclic vectors and of \mathcal{U} -frequently hypercyclic vectors are due to Bayart and Grivaux [7] and to Shkarin [38], respectively. For a set A of non-negative integers, the quantities

$$\underline{\text{dens}}(A) := \liminf_N \frac{\#(A \cap [0, N])}{N+1} \text{ and } \overline{\text{dens}}(A) := \limsup_N \frac{\#(A \cap [0, N])}{N+1}$$

are respectively called the lower density and the upper density of A .

Definition 1.5. Let (T_n) be a sequence in $L(X, Y)$. A vector $x \in X$ is said to be *frequently hypercyclic* (respectively, *\mathcal{U} -frequently hypercyclic*) for (T_n) provided for each non-empty open subset U of Y , the set $\{n \geq 0 : T_n x \in U\}$ has positive lower density (respectively, positive upper density).

Bonilla and Grosse-Erdmann [17, 18] showed that each non-scalar convolution operator $\phi(D)$ on $H(\mathbb{C})$ is frequently hypercyclic and that it has a frequently hypercyclic subspace whenever $\phi \in H(\mathbb{C})$ is transcendental, and asked whether the derivative operator D has a frequently hypercyclic subspace as well [18, Problem 2]. We will consider here the following related question.

Question 1. Does the derivative operator have a \mathcal{U} -frequently hypercyclic subspace on $H(\mathbb{C})$? What about $P(D)$, where P is a non-constant polynomial?

The next question is motivated by recent work of Bayart and Ruzsa [9], who completely characterized frequent hypercyclicity and \mathcal{U} -frequent hypercyclicity among unilateral and bilateral weighted shift operators on ℓ^p and $\ell^p(\mathbb{Z})$ ($1 \leq p < \infty$), respectively, as well as on c_0 and $c_0(\mathbb{Z})$, respectively. Hence it is natural to ask:

Question 2. Which weighted shifts support a \mathcal{U} -frequently hypercyclic subspace?

The second theme we consider is common hypercyclic subspaces. Recall the following.

Definition 1.6. A vector $x \in X$ is called a *common hypercyclic vector* for a given family $\mathcal{F} = \{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ of sequences in $L(X, Y)$ provided x is a hypercyclic vector for each $(T_{n,\lambda})_{n \geq 0}$ in \mathcal{F} . We also say that the common hypercyclic vectors of a given family $\mathcal{F} = \{T_\lambda\}_{\lambda \in \Lambda}$ of operators on X are the common hypercyclic vectors of the family of corresponding sequences of iterates $\{(T_\lambda^n)_{n \geq 0}\}_{\lambda \in \Lambda}$. A closed, infinite-dimensional subspace consisting entirely (but the origin) of common hypercyclic vectors for \mathcal{F} is called a *common hypercyclic subspace* for \mathcal{F} . A dense, linear subspace consisting entirely (but the origin) of common hypercyclic vectors for \mathcal{F} is called a *common hypercyclic manifold* for \mathcal{F} .

If the family $\mathcal{F} = \{T_\lambda\}_{\lambda \in \Lambda}$ of hypercyclic operators is countable, then it has a residual set of common hypercyclic vectors and a common hypercyclic manifold [27]. If \mathcal{F} is uncountable, however, it may fail to have a single common hypercyclic vector, even if each operator T_λ has a hypercyclic subspace [1]. The first example of an uncountable family with a common hypercyclic vector was given by Abakumov and Gordon [2], who showed that $\{\lambda B\}_{\lambda > 1}$ has a common hypercyclic vector, where B is the unweighted backward shift on ℓ^2 . The importance of common hypercyclic vectors within linear dynamics is showcased in [6, Chapter 7] and [28, Chapter 11].

In a remarkable paper, Costakis and Sambarino [22] considered unilateral backward shifts B_{w_λ} ($\lambda > 1$) on ℓ^2 with weight sequence $w_\lambda = (1 + \frac{\lambda}{n})_{n \geq 1}$ and showed that the set of common hypercyclic vectors for the family

$$\{B_{w_\lambda}\}_{\lambda > 1}$$

is residual on ℓ^2 . Each such B_{w_λ} has a hypercyclic subspace. Hence the following

Question 3. (Costakis and Sambarino [22, Problem 3]) Does the family $\{B_{w_\lambda}\}_{\lambda > 1}$ support a common hypercyclic subspace?

We already mentioned that any given non-scalar convolution operator $\phi(D)$ on $H(\mathbb{C})$ has a hypercyclic subspace. But the family of its (non-zero) scalar multiples

$$\{\lambda \phi(D)\}_{|\lambda| > 0}$$

has a residual set of common hypercyclic vectors, as established by Costakis and Mavroudis [21] for the case ϕ is a non-constant polynomial, by Bernal [11] for the case ϕ is of growth order no larger than $\frac{1}{2}$, and by Shkarin [39] for the general transcendental case. Hence it is natural to ask.

Question 4. Does the family of non-zero scalar multiples of a non-scalar convolution operator on $H(\mathbb{C})$ support a common hypercyclic subspace?

We mention that Bayart [4] extended Criterion M_0 to one that shows the existence of common hypercyclic subspaces (see Theorem 4.1), but this extension has proven difficult to apply in many natural examples.

1.3. Main results and outline of the paper. Section 2 is devoted to developing the main tools (Theorem 2.3) for establishing in the subsequent sections (M_k) -type criteria for the existence of \mathcal{U} -frequently hypercyclic subspaces or of common hypercyclic subspaces.

Section 3 is devoted to \mathcal{U} -frequently hypercyclic subspaces. We show for instance that any operator on a complex Banach space that satisfies the Frequent Hypercyclicity Criterion (Theorem 3.1) *has a \mathcal{U} -frequently hypercyclic subspace if and only if T has a hypercyclic subspace* (Theorem 3.8). We then answer Question 1 in the affirmative. Indeed, we show that *for any $N \geq 1$ and any $a_0(z), \dots, a_{N-1}(z) \in H(\mathbb{C})$ and $0 \neq a_N \in \mathbb{C}$ the linear differential operator*

$$L = a_N D^N + a_{N-1}(z) D^{N-1} + \dots + a_0(z) I$$

has a \mathcal{U} -frequently hypercyclic subspace on $H(\mathbb{C})$ (Corollary 3.14). Concerning Question 2, we show that *each \mathcal{U} -frequently hypercyclic unilateral weighted backward shift with a hypercyclic subspace on ℓ^p has a \mathcal{U} -frequently hypercyclic subspace* (Corollary 3.9). As a consequence, we derive the existence of *a frequently hypercyclic operator that has a \mathcal{U} -frequently hypercyclic subspace but no frequently hypercyclic subspace* (Corollary 3.10). We complement our answer to Question 2 by showing that each \mathcal{U} -frequently hypercyclic bilateral weighted backward shift with a hypercyclic subspace on $\ell^p(\mathbb{Z})$ has even a frequently hypercyclic subspace (Theorem 3.11).

We obtain the above results after establishing the following criterion for supporting \mathcal{U} -frequently hypercyclic subspaces (cf. Theorem 3.3 and Theorem 3.4): *If the Fréchet space X supports a continuous norm then any $T \in L(X)$ satisfying*

the Frequent Hypercyclicity Criterion has a \mathcal{U} -frequently hypercyclic subspace provided there exists a strictly increasing sequence of positive integers (n_k) with positive upper density satisfying either of the two conditions:

- (a) (T^{n_k}) converges pointwise to zero on some closed and infinite-dimensional subspace M_0 of X , or
- (b) (T^{n_k}) is equicontinuous along some non-increasing sequence (M_k) of closed infinite-dimensional subspaces of X .

Section 4 is devoted to common hypercyclic subspaces. We provide with Theorem 4.1 a constructive proof of a Criterion M_0 due to Bayart [4] for the existence of common hypercyclic subspaces but on Fréchet spaces that support a continuous norm, and use it to establish a corresponding Criterion (M_k) which is simpler to apply (Theorem 4.2). Indeed, we use the latter to answer Question 3 in the affirmative (Corollary 4.8) and to partially answer Question 4: *for any $N \geq 1$ and any $a_0(z), \dots, a_{N-1}(z) \in H(\mathbb{C})$, the family*

$$\{\lambda L\}_{0 \neq \lambda \in \mathbb{C}}$$

of non-zero scalar multiples of the linear differential operator

$$L = D^n + a_{N-1}(z)D^{N-1} + \dots + a_0(z)I$$

has a common hypercyclic subspace on $H(\mathbb{C})$ (Corollary 4.12). Finally, we seek in the last subsection spectral characterizations for the existence of common hypercyclic subspaces for certain families of operators, complementing the spectral characterization by González, León and Montes [26] for the case of a single operator. In particular, we show that for a complex Banach space X and $0 < a < b \leq \infty$, a family of non-zero scalar multiples of a given $T \in L(X)$

$$\{\lambda T\}_{a < \lambda < b}$$

that satisfies the Common Hypercyclicity Criterion has a common hypercyclic subspace if and only if the essential spectrum of T contains an element of modulus no larger than $\frac{1}{b}$, with the convention that $\frac{1}{b} = 0$ when $b = \infty$ (Corollary 4.16 and Corollary 4.17).

We finish this introduction with a brief subsection on the Common Hypercyclicity Criterion, which we use in Section 4.

1.4. The Common Hypercyclicity Criterion. Several criteria have been used to show the existence of common hypercyclic vectors thanks to the works of Costakis and Sambarino [22], Bayart and Matheron [5], and others [3, 8, 11, 21, 24, 28]. In this paper we use the following version of the Common Hypercyclicity Criterion based on [28, Remark 11.10].

Definition 1.7 (Common Hypercyclicity Criterion). Let $\Lambda \subseteq \mathbb{R}$ be an open interval. We say that a family $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ of sequences in $L(X, Y)$ satisfies the *Common Hypercyclicity Criterion* (CHC) provided for each $(n, x) \in \mathbb{Z}^+ \times X$ the vector $T_{n,\lambda}x$ depends continuously on $\lambda \in \Lambda$ and provided for each compact subset K of Λ there exist dense subsets X_0 and Y_0 of X and Y , respectively, and mappings

$$S_{n,\lambda} : Y_0 \rightarrow X \quad (\lambda \in K, n = 0, 1, \dots)$$

so that for each $y_0 \in Y_0$, each $x_0 \in X_0$, we have

- (1) $\sum_{k=0}^m T_{m,\lambda} S_{m-k,\mu_k} y_0$ converges unconditionally and uniformly on $\lambda \geq \mu_0 \geq \mu_1 \geq \dots \geq \mu_m$ in K and $m \geq 0$.

- (2) $\sum_{k=0}^{\infty} T_{m,\lambda} S_{m+k,\mu_k} y_0$ converges unconditionally and uniformly on $\lambda \leq \mu_0 \leq \mu_1 \leq \dots$ in K and $m \geq 0$.
- (3) For each $\varepsilon > 0$ and each continuous seminorm q on Y , there exists a sequence (δ_k) of positive real numbers such that $\sum_{k=0}^{\infty} \delta_k = \infty$ and for each $k \geq 0$ and $\lambda, \mu \in K$ we have

$$0 \leq \mu - \lambda < \delta_k \Rightarrow q(T_{k,\lambda} S_{k,\mu} y_0 - y_0) < \varepsilon.$$

- (4) $T_{k,\lambda} x_0 \xrightarrow[k \rightarrow \infty]{} 0$ uniformly on $\lambda \in K$.
- (5) $\sum_{k=0}^{\infty} S_{k,\mu_k} y_0$ converges unconditionally and uniformly on $\mu_0 \leq \mu_1 \leq \dots$ in K .

Theorem 1.8. *Let $\{(T_{n,\lambda})_n\}_{\lambda \in \Lambda}$ be a family of sequences in $L(X, Y)$ satisfying (CHC). Then the set of vectors in X that are hypercyclic for every sequence $(T_{n,\lambda})_n$ is residual in X .*

Theorem 1.8 (follows from the same arguments and) slightly generalizes the Common Hypercyclicity Criterion given in [28, Theorem 11.9, Remark 11.10(d)].

Remark 1.9.

- (a) If $\{T_\lambda\}_{\lambda \in \Lambda}$ is a family of operators satisfying (the assumptions of) the Common Hypercyclicity Criterion of Costakis and Sambarino [22], then the family $\{(T_\lambda^{n+1})_{n \geq 0}\}_{\lambda \in \Lambda}$ satisfies (CHC) as defined above in Definition 1.7.
- (b) Given a scalar $\lambda_0 \geq 0$ and $T \in L(X)$, the family $\{(\lambda^n T^n)_{n \geq 1}\}_{\lambda > \lambda_0}$ satisfies (CHC) if there exists a dense subset A of X that is contained in $\bigcup_{n \geq 1} \text{Ker}(T^n)$ and maps $S_n : A \rightarrow X$ so that for each $x \in A$ we have that (i) $T^n S_n x = x$, that (ii) $T^m S_{m+n} x = S_n x$ for each $m, n \geq 0$, and that (iii) $\{\frac{1}{\lambda^n} S_n x\}_{n \geq 1}$ is bounded in X for each $\lambda > \lambda_0$, see [28, Section 11.2].

2. MAIN TOOL FOR THE GENERALIZATION OF CRITERION (M_k)

In this section we develop a link between Criterion M_0 and Criterion (M_k) . Notice that Criterion M_0 immediately gives Criterion (M_k) , by the Banach-Steinhaus theorem. Conversely, if a sequence of operators (T_n) satisfies Criterion (M_k) along a given sequence of integers (n_k) , then it also satisfies Criterion M_0 for some subsequence (m_k) of (n_k) [32]. The section's main result, Theorem 2.3, allows us to get this subsequence (m_k) to inherit special properties of the original sequence (n_k) (see Lemma 3.5 and Proposition 4.3) which is a key ingredient for establishing (M_k) -type criteria for the existence of \mathcal{U} -frequently hypercyclic subspaces or common hypercyclic subspaces from the corresponding M_0 -type criteria. We need the following two definitions.

Definition 2.1. We say that $(\Lambda_n)_{n \geq 1}$ is a *chain* of a given set Λ if the sequence $(\Lambda_n)_{n \geq 1}$ is non-decreasing and $\bigcup_{n \geq 1} \Lambda_n = \Lambda$.

Definition 2.2. Let (M_n) be a sequence of infinite-dimensional closed subspaces of X with $M_n \supseteq M_{n+1}$ for all n . A family $\{(T_{n,\lambda})\}_{\lambda \in \Lambda}$ of sequences in $L(X, Y)$ is said to be *uniformly equicontinuous along (M_n)* provided for each continuous seminorm q on Y there exists a continuous seminorm p of X so that for each $n \in \mathbb{N}$

$$\sup_{\lambda \in \Lambda} q(T_{n,\lambda} x) \leq p(x) \quad \text{for each } x \in M_n.$$

Theorem 2.3. *Let X be an infinite-dimensional Fréchet space with continuous norm, let Y be a separable Fréchet space and let Λ be a set. Let $\{(T_{k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda}$ be a family of sequences of operators in $L(X, Y)$. Suppose that there exist chains $(\Lambda_n^j)_{n \geq 1}$ of Λ ($j = 0, 1, 2$) satisfying:*

- (i) *for each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$, the family $\{T_{k,\lambda}\}_{\lambda \in \Lambda_n^0}$ is equicontinuous;*
- (ii) *for each $n \in \mathbb{N}$, there exists a dense subset $X_{n,0}$ of X such that for any $x \in X_{n,0}$,*

$$T_{k,\lambda}x \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{uniformly on } \lambda \in \Lambda_n^1;$$

- (iii) *there exists a non-increasing sequence of infinite-dimensional closed subspaces (M_k) of X such that for each $n \geq 1$, the family of sequences*

$$\{(T_{k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda_n^2}$$

is uniformly equicontinuous along (M_k) .

Then for any map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ there exist an increasing sequence of integers $(k_s)_{s \geq 1}$ and an infinite-dimensional closed subspace M_0 of X such that for any $(x, \lambda) \in M_0 \times \Lambda$,

$$T_{k,\lambda}x \xrightarrow[k \rightarrow \infty]{k \in I} 0,$$

where $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$.

The choice of the function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ in Theorem 2.3 is the main tool to generalize Criterion (M_k) to \mathcal{U} -frequently hypercyclic subspaces (Section 3) and to common hypercyclic subspaces (Section 4). The proof of Theorem 2.3 relies on the notion of a basic sequence, which has been commonly used for the construction of closed, infinite-dimensional subspaces whose vectors are to satisfy special properties. We refer to [32, 36] for more details about the construction of basic sequences in Fréchet spaces with a continuous norm.

Definition 2.4. A sequence $(u_k)_{k \geq 1}$ in a Fréchet space is called *basic* if for every $x \in \overline{\text{span}}\{u_k : k \geq 1\}$, there exists a unique sequence $(a_k)_{k \geq 1}$ in \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) such that $x = \sum_{k=1}^{\infty} a_k u_k$.

Proof of Theorem 2.3. Let (p_j) be an increasing sequence of norms inducing the topology of X and let (q_j) be an increasing sequence of seminorms inducing the topology of Y . We consider $\Lambda_n = \Lambda_n^0 \cap \Lambda_n^1 \cap \Lambda_n^2$ and a basic sequence $(u_i)_{i \geq 1}$ in X such that $u_i \in M_i$ and $p_1(u_i) = 1$ for each $i \geq 1$, and so that for every $j \geq 1$, the sequence $(u_i)_{i \geq j}$ is basic in $X_j := (X, p_j)$ with basic constant less than 2. By (i), for each $n, k, j \geq 1$, there exists $K_{n,k,j} > 0$ and $m^0(n, k, j) \in \mathbb{N}$ such that for any $x \in X$,

$$(2.1) \quad \sup_{\lambda \in \Lambda_n} q_j(T_{k,\lambda}x) \leq K_{n,k,j} p_{m^0(n,k,j)}(x).$$

By (iii), for each $n, j \geq 1$ there exist $C_{n,j} > 0$ and $m(n, j) \in \mathbb{N}$ such that for any $k \geq 1$ and any $x \in M_k$,

$$(2.2) \quad \sup_{\lambda \in \Lambda_n} q_j(T_{k,\lambda}x) \leq C_{n,j} p_{m(n,j)}(x).$$

Since each set $X_{i,0}$ is dense, we can construct a family $(x_{i,l})_{i,l \geq 1} \subset X$ and a sequence of integers $(k_l)_{l \geq 0}$ with $k_0 = 1$ such that for any $l \geq 1$,

(1) for any $i \geq 1$, any $n, k, j \leq i$,

$$(2.3) \quad p_{m^0(n,k,j)}(x_{i,l}) < \frac{1}{2^{i+l}K_{n,k,j}} \quad \text{and} \quad p_i(x_{i,l}) < \frac{1}{2^{i+l+2}};$$

(2) for any $i, k \leq k_{l-1} + \phi(k_{l-1})$,

$$(2.4) \quad \sup_{\lambda \in \Lambda_{k_{l-1} + \phi(k_{l-1})}} q_{k_{l-1} + \phi(k_{l-1})}(T_{k,\lambda} x_{i,l}) < \frac{1}{2^{i+l}};$$

(3) for any $i \geq 1$, $u_i + \sum_{l'=1}^l x_{i,l'} \in X_{k_{l-1} + \phi(k_{l-1}), 0}$;

(4) $k_l > k_{l-1} + \phi(k_{l-1})$;

(5) for any $i \leq k_{l-1} + \phi(k_{l-1})$, any $k \in [k_l, k_l + \phi(k_l)]$,

$$(2.5) \quad \sup_{\lambda \in \Lambda_{k_{l-1} + \phi(k_{l-1})}} q_{k_{l-1} + \phi(k_{l-1})} \left(T_{k,\lambda} \left(u_i + \sum_{l'=1}^l x_{i,l'} \right) \right) < \frac{1}{2^{l+i}}.$$

Indeed, in order to satisfy (1) and (2), it suffices to choose $x_{i,l}$ sufficiently small thanks to (2.1) and since we choose $x_{i,l}$ such that $u_i + \sum_{l'=1}^l x_{i,l'} \in X_{k_{l-1} + \phi(k_{l-1}), 0}$, we know that for any $i \geq 1$, $T_{k,\lambda} \left(u_i + \sum_{l'=1}^l x_{i,l'} \right)$ tends uniformly to 0 on $\lambda \in \Lambda_{k_{l-1} + \phi(k_{l-1})}$. We can thus find k_l sufficiently big such that (4) and (5) are satisfied.

For any $n \geq 1$, we let $x_n = u_n + \sum_{l=1}^{\infty} x_{n,l}$. We deduce from (2.3) that $(x_{k_l + \phi(k_l)})_{l \geq 1}$ is a basic sequence equivalent to the sequence $(u_{k_l + \phi(k_l)})_{l \geq 1}$. Let M_0 be the closed linear span of $(x_{k_l + \phi(k_l)})_{l \geq 1}$ in X and x a vector in M_0 . We know that we have $x = \sum_{s=1}^{\infty} a_s x_{k_s + \phi(k_s)}$ where the sequence $(a_s)_{s \geq 1}$ is bounded by some constant K .

Let $n, j \geq 1$ and $l \geq 2$ with $n, j \leq k_{l-1} + \phi(k_{l-1})$. Since

$$\sum_{s=l}^{\infty} a_s u_{k_s + \phi(k_s)} \in M_{k_l + \phi(k_l)},$$

we deduce from (2.1), (2.2), (2.3), (2.4) and (2.5) that for any $\lambda \in \Lambda_n$, any $k \in [k_l, k_l + \phi(k_l)]$,

$$\begin{aligned} q_j(T_{k,\lambda} x) &\leq \sum_{s=1}^{l-1} |a_s| q_j(T_{k,\lambda} x_{k_s + \phi(k_s)}) + \sum_{s=l}^{\infty} |a_s| q_j(T_{k,\lambda} (x_{k_s + \phi(k_s)} - u_{k_s + \phi(k_s)})) \\ &\quad + q_j \left(T_{k,\lambda} \left(\sum_{s=l}^{\infty} a_s u_{k_s + \phi(k_s)} \right) \right) \\ &\leq \sum_{s=1}^{l-1} |a_s| q_{k_{l-1} + \phi(k_{l-1})} \left(T_{k,\lambda} \left(u_{k_s + \phi(k_s)} + \sum_{l'=1}^l x_{k_s + \phi(k_s), l'} \right) \right) \\ &\quad + \sum_{s=1}^{l-1} \sum_{l'=l+1}^{\infty} |a_s| q_{k_{l'-1} + \phi(k_{l'-1})} (T_{k,\lambda} x_{k_s + \phi(k_s), l'}) \\ &\quad + \sum_{s=l}^{\infty} \sum_{l'=1}^{\infty} |a_s| q_j(T_{k,\lambda} x_{k_s + \phi(k_s), l'}) + q_j \left(T_{k,\lambda} \left(\sum_{s=l}^{\infty} a_s u_{k_s + \phi(k_s)} \right) \right) \\ &\leq \sum_{s=1}^{l-1} \frac{K}{2^{l+k_s + \phi(k_s)}} + \sum_{s=1}^{l-1} \sum_{l'=l+1}^{\infty} \frac{K}{2^{k_s + \phi(k_s) + l'}} \end{aligned}$$

$$\begin{aligned}
& + K \sum_{s=l}^{\infty} \sum_{l'=1}^{\infty} K_{n,k,j} p_{m^0(n,k,j)}(x_{k_s+\phi(k_s),l'}) + C_{n,j} p_{m(n,j)} \left(\sum_{s=l}^{\infty} a_s u_{k_s+\phi(k_s)} \right) \\
& \leq \frac{lK}{2^{l-1}} + K \sum_{s=l}^{\infty} \frac{K_{n,k,j}}{2^{k_s+\phi(k_s)} K_{n,k,j}} + C_{n,j} p_{m(n,j)} \left(\sum_{s=l}^{\infty} a_s u_{k_s+\phi(k_s)} \right) \\
& \leq \frac{lK}{2^{l-1}} + K \sum_{s=l}^{\infty} \frac{1}{2^{k_s+\phi(k_s)}} + C_{n,j} p_{m(n,j)} \left(\sum_{s=l}^{\infty} a_s u_{k_s+\phi(k_s)} \right) \xrightarrow{l \rightarrow \infty} 0.
\end{aligned}$$

□

We finish this section by stating two particular cases of Theorem 2.3. In Corollary 2.5 below we consider the case when $\{T_\lambda\}_{\lambda \in \Lambda}$ is a family of operators on a Banach space X . We use Corollary 2.5 in Section 4.3 to study the existence of common hypercyclic subspaces for collections of scalar multiples of a fixed operator on a complex Banach space.

Corollary 2.5. *Let $(X, \|\cdot\|)$ be a separable infinite-dimensional Banach space, let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a family of operators in $L(X)$, and let $(n_k)_{k \geq 1}$ be a strictly increasing sequence of positive integers. Suppose there exist chains (Λ_n^1) and (Λ_n^2) of Λ satisfying:*

- (i) *for any $n \geq 1$, there exists a dense subset $X_{n,0}$ of X such that for any $x \in X_{n,0}$,*

$$\sup_{\lambda \in \Lambda_n^1} \|T_\lambda^{n_k} x\| \xrightarrow{k \rightarrow \infty} 0;$$

- (ii) *there exists a non-increasing sequence (M_k) of infinite-dimensional closed subspaces of X such that for any $n \in \mathbb{N}$,*

$$\sup_{(\lambda,k) \in \Lambda_n^2 \times \mathbb{N}} \|T_\lambda^{n_k}\|_{M_k} < \infty.$$

Then for any $\phi : \mathbb{N} \rightarrow \mathbb{N}$, there exist an increasing sequence of integers $(k_s)_{s \geq 1}$ and an infinite-dimensional closed subspace M_0 of X such that for any $x \in M_0$ and any $\lambda \in \Lambda$,

$$T_\lambda^{n_k} x \xrightarrow[k \rightarrow \infty]{k \in I} 0$$

where $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$.

Proof. In view of Theorem 2.3, it suffices to prove the existence of a chain (Λ_n^0) of Λ such that for any $n, k \geq 1$, there exists $K_{n,k}$ such that for any $x \in X$,

$$\sup_{\lambda \in \Lambda_n^0} \|T_\lambda^{n_k} x\| \leq K_{n,k} \|x\|.$$

For each $n \in \mathbb{N}$, let $\Lambda_n^0 = \{\lambda \in \Lambda : \|T_\lambda\| \leq n\}$. So (Λ_n^0) is a chain of Λ and for any $n, k \in \mathbb{N}$ and $x \in X$, we have

$$\sup_{\lambda \in \Lambda_n^0} \|T_\lambda^{n_k} x\| \leq \sup_{\lambda \in \Lambda_n^0} \|T_\lambda^{n_k}\| \|x\| \leq \sup_{\lambda \in \Lambda_n^0} \|T_\lambda\|^{n_k} \|x\| \leq n^{n_k} \|x\|.$$

We conclude that the chain (Λ_n^0) satisfied the desired inequalities for $K_{n,k} = n^{n_k}$. □

Finally, for the case of a single sequence of operators (T_n) in $L(X, Y)$, Theorem 2.3 gives the following.

Corollary 2.6. *Let X be an infinite-dimensional Fréchet space with a continuous norm, let Y be a separable Fréchet space and let (T_n) be a sequence in $L(X, Y)$. Suppose there exists a strictly increasing sequence of integers (n_k) satisfying*

- (a) $T_{n_k}x \xrightarrow[k \rightarrow \infty]{} 0$ for each x in some dense subset X_0 of X , and
- (b) $(T_{n_k})_{k \geq 1}$ is equicontinuous along some non-increasing sequence (M_k) of infinite-dimensional closed subspaces of X .

Then for any $\phi : \mathbb{N} \rightarrow \mathbb{N}$ there exist an increasing sequence of integers $(k_s)_{s \geq 1}$ and an infinite-dimensional closed subspace M_0 such that if $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$, then for any $x \in M_0$,

$$T_{n_k}x \xrightarrow[k \rightarrow \infty]{k \in I} 0.$$

We use Corollary 2.6 in Section 3 to obtain a version of Criterion (M_k) for the existence of \mathcal{U} -frequently hypercyclic subspaces.

3. EXISTENCE OF \mathcal{U} -FREQUENTLY HYPERCYCLIC SUBSPACES

The existence of frequently hypercyclic subspaces was first investigated by Bonilla and Grosse-Erdmann [18], who generalized Criterion M_0 to one for obtaining frequently hypercyclic subspaces, which has in turn been applied to convolution operators and weighted composition operators, see [18], [12], [13]. We first recall the Frequent Hypercyclicity Criterion due to Bayart and Grivaux [7]; the version we use here is due to Bonilla and Grosse-Erdmann [17].

Theorem 3.1 (Frequent Universality Criterion). *Let X be a Fréchet space, let Y be a separable Fréchet space and let (T_n) be a sequence in $L(X, Y)$. If there exist a dense subset $Y_0 \subset Y$ and $S_k : Y_0 \rightarrow X$, $k \geq 0$ such that for each $y \in Y_0$,*

1. $\sum_{n=0}^{\infty} S_n y$ converges unconditionally in X ,
2. $\sum_{n=1}^k T_k S_{k-n} y$ converges unconditionally in Y , uniformly in $k \geq 0$,
3. $\sum_{n=1}^{\infty} T_k S_{k+n} y$ converges unconditionally in Y , uniformly in $k \geq 0$,
4. $T_n S_n y \rightarrow y$,

then (T_n) is frequently hypercyclic.

We can now state (a slightly more general formulation of) the Bonilla and Grosse-Erdmann's criterion for the existence of frequently hypercyclic subspaces.

Theorem 3.2. (Bonilla and Grosse-Erdmann) *Let X be a Fréchet space with a continuous norm and let Y be a separable Fréchet space. Let (T_n) be a sequence in $L(X, Y)$ satisfying the Frequent Universality Criterion. Suppose that*

$$T_n x \xrightarrow[n \rightarrow \infty]{} 0$$

for each vector x in some closed, infinite-dimensional subspace M_0 of X . Then (T_n) has a frequently hypercyclic subspace.

Under the assumption that (T_n) satisfies the Frequent Universality Criterion, their proof of the above theorem (see [18, Theorem 3]) can then be adapted to \mathcal{U} -frequently hypercyclic subspaces as follows:

Theorem 3.3 (Criterion M_0 for \mathcal{U} -Frequently Hypercyclic Subspaces). *Let X be a Fréchet space with a continuous norm and let Y be a separable Fréchet space. Let (T_n) be a sequence in $L(X, Y)$ satisfying the Frequent Universality Criterion*

and for which there exist an infinite-dimensional closed subspace M_0 and a strictly increasing sequence $(n_k)_{k \geq 1}$ of positive upper density such that for any $x \in M_0$,

$$T_{n_k}x \xrightarrow[k \rightarrow \infty]{} 0.$$

Then (T_n) has a \mathcal{U} -frequently hypercyclic subspace.

Proof. By applying Theorem 3.2 to the sequence of operators $(T_{n_k})_{k \geq 1}$, we obtain an infinite-dimensional closed subspace M such that for any non-zero vector $x \in M$ and any non-empty open set U in Y the return set $\{k \geq 1 : T_{n_k}x \in U\}$ is a set of positive lower density. Since $(n_k)_{k \geq 1}$ is a set of positive upper density, the set $(n_k)_{k \in A}$ is a set of positive upper density for any set A of positive lower density. Indeed, we have

$$\frac{\#(\{n_k : k \in A\} \cap [0, N])}{N+1} = \frac{\#(A \cap [1, j])}{j} \frac{\#(\{n_k : k \in \mathbb{N}\} \cap [0, N])}{N+1},$$

where $j = \#(\{n_k : k \in \mathbb{N}\} \cap [0, N])$. We conclude that M is a \mathcal{U} -frequently hypercyclic subspace. \square

We next derive the following (M_k) -criterion for the existence of \mathcal{U} -frequently hypercyclic subspaces, which is simpler to apply than Theorem 3.3.

Theorem 3.4 (Criterion (M_k) for \mathcal{U} -Frequently Hypercyclic Subspaces).

Let X be an infinite-dimensional Fréchet space with a continuous norm, let Y be a separable Fréchet space and let (T_n) be a sequence in $L(X, Y)$ satisfying the Frequent Universality Criterion. Suppose that there exists a strictly increasing sequence (n_k) of positive upper density so that

- (1) $T_{n_k}x \xrightarrow[k \rightarrow \infty]{} 0$ for each x in some dense subset X_0 of X , and
- (2) $(T_{n_k})_{k \geq 1}$ is equicontinuous along some non-increasing sequence (M_k) of infinite-dimensional closed subspaces of X .

Then (T_n) has a \mathcal{U} -frequently hypercyclic subspace.

We will prove Theorem 3.4 with Theorem 3.3 and by applying Corollary 2.6 with a suitable map ϕ whose existence is given by the following lemma.

Lemma 3.5. Let (n_k) be a strictly increasing sequence in \mathbb{N} with positive upper density. Then there exists a map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ so that for every strictly increasing sequence (k_s) in \mathbb{N} we have

$$\overline{\text{dens}}(\{n_k : k \in \cup_{s \geq 1} [k_s, k_s + \phi(k_s)]\}) = \overline{\text{dens}}(n_k).$$

Proof. Let $\delta = \overline{\text{dens}}(n_k)$, and choose $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $k \geq 1$,

$$\frac{\phi(k) + 1}{n_k + \phi(k)} \geq \delta - \frac{\delta}{k}.$$

Such a map ϕ exists because for any $k \geq 1$,

$$\delta = \limsup_N \frac{\#\{j : n_j \in [n_k, N]\}}{N} = \limsup_l \frac{\#\{j : n_j \in [n_k, n_l]\}}{n_l} = \limsup_l \frac{l+1-k}{n_l}$$

and thus for any $k \geq 1$ there exists $\phi(k)$ such that

$$\frac{\phi(k) + 1}{n_k + \phi(k)} \geq \delta - \frac{\delta}{k}.$$

The upper density of the subsequence $(m_l) = \{n_k : k \in \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]\}$ of (n_k) equals δ . Indeed, $\overline{\text{dens}}(m_l) \leq \overline{\text{dens}}(n_k) = \delta$ since (m_l) is a subsequence of (n_k) , and on the other hand

$$\begin{aligned} \overline{\text{dens}}(m_l) &= \limsup_N \frac{\#\{j : m_j \in [0, N]\}}{N} \\ &\geq \limsup_s \frac{\#\{j : m_j \in [n_{k_s}, n_{k_s + \phi(k_s)}]\}}{n_{k_s + \phi(k_s)}} \\ &= \limsup_s \frac{\phi(k_s) + 1}{n_{k_s + \phi(k_s)}} \geq \limsup_s \left(\delta - \frac{\delta}{k_s} \right) = \delta. \end{aligned}$$

So Lemma 3.5 follows. \square

Proof of Theorem 3.4. Let ϕ as in Lemma 3.5. By Corollary 2.6, there exists a closed infinite-dimensional subspace M_0 of X and a strictly increasing sequence (k_s) in \mathbb{N} such that for any $x \in M_0$,

$$T_{n_k} x \xrightarrow[k \rightarrow \infty]{k \in I} 0$$

where $I = \bigcup_{s \geq 0} [k_s, k_s + \phi(k_s)]$. Since (n_k) is a sequence of positive upper density, the subsequence $(m_l) = \{n_k : k \in I\}$ has positive upper density (Lemma 3.5) and the conclusion now follows from Theorem 3.3. \square

If we consider an operator $T \in L(X)$ and its iterates, we can consider the following version of the Frequent Universality Criterion.

Theorem 3.6 (Frequent Hypercyclicity Criterion). *Let X be a separable Fréchet space and $T \in L(X)$. If there are a dense subset X_0 of X and $S_k : X_0 \rightarrow X$, $k \geq 1$, such that for each $x \in X_0$,*

1. $\sum_{n=1}^{\infty} T^n x$ converges unconditionally,
2. $\sum_{n=1}^{\infty} S_n x$ converges unconditionally,
3. $T^m S_n x = S_{n-m} x$ for any $m < n$, and
4. $T^n S_n x = x$ for any $n \geq 1$.

Then T is frequently hypercyclic.

We remark that if T satisfies the Frequent Hypercyclicity Criterion, then its sequence of iterates (T^n) satisfies the Frequent Universality Criterion and there exists a dense subset X_0 of X such that $T^n x \rightarrow 0$ for each $x \in X_0$. Therefore, we obtain the following version of the Criterion (M_k) for \mathcal{U} -frequently hypercyclic subspaces.

Theorem 3.7. *Let X be a separable infinite-dimensional Fréchet space with a continuous norm and $T \in L(X)$ an operator satisfying the Frequent Hypercyclicity Criterion. Suppose that there exists a strictly increasing sequence of positive integers (n_k) of positive upper density such that $(T^{n_k})_{k \geq 1}$ is equicontinuous along some non-increasing sequence (M_k) of infinite-dimensional closed subspaces of X . Then T possesses a \mathcal{U} -frequently hypercyclic subspace.*

In 2000, González, León and Montes [26] obtained a characterization of operators with hypercyclic subspaces on complex Banach spaces. In particular, we can deduce from their proof that if T is an operator on a complex Banach space satisfying the Hypercyclicity Criterion, then T possesses a hypercyclic subspace if

and only if $(T^k)_{k \geq 0}$ is equicontinuous along some non-increasing sequence (M_k) of infinite-dimensional closed subspaces of X , see [26, Theorem 3.2]. In view of Theorem 3.7 and of the characterization of González, León and Montes, we conclude the following.

Theorem 3.8. *Let X be a separable infinite-dimensional complex Banach space and $T \in L(X)$ be an operator satisfying the Frequent Hypercyclicity Criterion. Then T has a hypercyclic subspace if and only if it has a \mathcal{U} -frequently hypercyclic subspace.*

Theorem 3.7 also gives the following characterization of unilateral weighted shifts with \mathcal{U} -frequently hypercyclic subspaces on real or complex Banach spaces ℓ^p ($1 \leq p < \infty$).

Corollary 3.9. *Let $B_w : \ell^p \rightarrow \ell^p$ be a \mathcal{U} -frequently hypercyclic unilateral weighted shift. Then B_w has a \mathcal{U} -frequently hypercyclic subspace if and only if B_w has a hypercyclic subspace. Thus a unilateral backward shift B_w with weight sequence $w = (w_n)$ has a \mathcal{U} -frequently hypercyclic subspace if and only if $(\frac{1}{w_1 \cdots w_n}) \in \ell^p$ and $\sup_{n \geq 1} \inf_{k \geq 0} \prod_{\nu=1}^n |w_{k+\nu}| \leq 1$.*

Proof. It is immediate that B_w has a hypercyclic subspace if it has a \mathcal{U} -frequently hypercyclic subspace. Conversely, if B_w has a hypercyclic subspace we can deduce from [29, 32] that there exists a non-increasing sequence of infinite-dimensional closed subspaces (M_k) in X such that

$$\sup_{k \geq 1} \|B_w^k|_{M_k}\| < \infty.$$

We also know thanks to Bayart and Ruzsa [9] that if $B_w : \ell^p \rightarrow \ell^p$ is a \mathcal{U} -frequently hypercyclic weighted shift, then B_w satisfies the Frequent Hypercyclicity Criterion. We therefore conclude by Theorem 3.7 that B_w has a \mathcal{U} -frequently hypercyclic subspace. The second assertion follows now from the facts that B_w is \mathcal{U} -frequently hypercyclic on ℓ^p if and only if $(\frac{1}{w_1 \cdots w_n}) \in \ell^p$ [9] and that B_w has a hypercyclic subspace if and only if $\sup_{n \geq 1} \inf_{k \geq 0} \prod_{\nu=1}^n |w_{k+\nu}| \leq 1$ [29, 32]. \square

This result may seem surprising given that a weighted shift on ℓ^p is \mathcal{U} -frequently hypercyclic if and only if it is frequently hypercyclic [9], and there exist frequently hypercyclic weighted shifts with hypercyclic subspaces and no frequently hypercyclic subspace [34]. In particular, we deduce the following.

Corollary 3.10. *There exists a frequently hypercyclic operator that has \mathcal{U} -frequently hypercyclic subspaces but has no frequently hypercyclic subspace.*

The above results motivate the following.

Problem 1. : Does there exist a \mathcal{U} -frequently hypercyclic operator possessing hypercyclic subspaces but no \mathcal{U} -frequently hypercyclic subspace? What if the operator is frequently hypercyclic?

We complement our study of the existence of \mathcal{U} -frequently hypercyclic subspaces for weighted shifts by showing that every \mathcal{U} -frequently hypercyclic bilateral weighted shift on $\ell^p(\mathbb{Z})$ possesses a frequently hypercyclic subspace and thus a \mathcal{U} -frequently hypercyclic subspace.

Theorem 3.11. *Let B_w be a bilateral weighted shift on $\ell^p(\mathbb{Z})$. If B_w is \mathcal{U} -frequently hypercyclic, then B_w possesses a frequently hypercyclic subspace.*

Proof. Let B_w be a \mathcal{U} -frequently hypercyclic bilateral weighted shift on $\ell^p(\mathbb{Z})$. We already know that B_w satisfies the Frequent Hypercyclicity Criterion [9]. In view of Theorem 3.3, it thus suffices to prove that there exists an infinite-dimensional closed subspace M_0 such that for any $x \in M_0$

$$B_w^n x \xrightarrow{n \rightarrow \infty} 0.$$

Since B_w is \mathcal{U} -frequently hypercyclic, we also know that $\sum_{n \leq 0} |w_0 \cdots w_n|^p < \infty$ [9]. In particular, we have for any $k \geq 1$

$$(3.1) \quad \prod_{\nu=0}^n |w_{-k-\nu}| = \frac{\prod_{\nu=0}^{k+n} |w_{-\nu}|}{\prod_{\nu=0}^{k-1} |w_{-\nu}|} \xrightarrow{n \rightarrow \infty} 0.$$

Let $k_0 \geq 1$. We show that there exists $k_1 \geq k_0$ such that for any $n \geq 0$,

$$\prod_{\nu=0}^n |w_{-k_1-\nu}| \leq 1.$$

Indeed, either for any $n \geq 0$, we have $\prod_{\nu=0}^n |w_{-k_0-\nu}| \leq 1$ and we can consider $k_1 = k_0$, or the set $F := \{n \geq 0 : \prod_{\nu=0}^n |w_{-k_0-\nu}| > 1\}$ is non-empty. We then remark that if $n \in F$, we have

$$\prod_{\nu=0}^{k_0+n} |w_{-\nu}| = \left(\prod_{\nu=0}^n |w_{-k_0-\nu}| \right) \left(\prod_{\nu=0}^{k_0-1} |w_{-\nu}| \right) > \prod_{\nu=0}^{k_0-1} |w_{-\nu}|.$$

We deduce from (3.1) that F has to be finite. Let $n_0 := \max F$ and $k_1 = k_0 + n_0 + 1$. We then have $k_1 \geq k_0$ and for any $n \geq 0$,

$$\prod_{\nu=0}^n |w_{-k_1-\nu}| = \frac{\prod_{\nu=0}^{n_0+n+1} |w_{-k_0-\nu}|}{\prod_{\nu=0}^{n_0} |w_{-k_0-\nu}|} \leq 1.$$

We conclude that there exists an increasing sequence of positive integers $(k_j)_{j \geq 0}$ such that for any $j \geq 0$, any $n \geq 0$,

$$(3.2) \quad \prod_{\nu=0}^n |w_{-k_j-\nu}| \leq 1.$$

Let $M_0 := \overline{\text{span}}\{e_{-k_j} : j \geq 0\}$ and $x \in M_0$. We have $x = \sum_{j=0}^{\infty} a_j e_{-k_j}$ and for any $n \geq 1$, any $J \geq 0$, we deduce from (3.1) and (3.2) that

$$\begin{aligned} \|B_w^n x\|^p &= \sum_{j=0}^{\infty} \left(\prod_{\nu=0}^{n-1} |w_{-k_j-\nu}| \right)^p |a_j|^p \\ &\leq \sum_{j=0}^J \left(\prod_{\nu=0}^{n-1} |w_{-k_j-\nu}| \right)^p |a_j|^p + \sum_{j=J+1}^{\infty} |a_j|^p \xrightarrow{n \rightarrow \infty} \sum_{j=J+1}^{\infty} |a_j|^p. \end{aligned}$$

We thus have $\|B_w^n x\| \rightarrow 0$ for any $x \in M_0$ and we conclude by using Theorem 3.3. \square

We finish this section by investigating the operators $P(D)$ on the space $H(\mathbb{C})$ of entire functions where P is a non-constant polynomial and D is the derivative operator.

Corollary 3.12. *Let $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ be the derivative operator and P a non-constant polynomial. Then $P(D)$ has a \mathcal{U} -frequently hypercyclic subspace.*

Proof. We know that $P(D)$ satisfies the Frequently Hypercyclicity Criterion [16]. It is also shown in [32] that if $P(D)$ satisfies the Hypercyclicity Criterion along a given strictly increasing sequence (n_k) , then there exists a non-increasing sequence of infinite-dimensional closed subspaces (M_k) in $H(\mathbb{C})$ such that for any $j \geq 1$, there exist a positive number C_j and two integers $m(j), k(j)$ such that for any $k \geq k(j)$, any $x \in M_k$,

$$p_j(P(D)^{n_k}x) \leq C_j p_{m(j)}(x).$$

In other words, it is shown that $(P(D)^{n_k})$ is equicontinuous along (M_k) . Since $P(D)$ satisfies the Frequent Hypercyclicity Criterion, $P(D)$ satisfies the Hypercyclicity Criterion along the whole sequence (k) and we can conclude by applying Theorem 3.7 along the whole sequence $(n_k) = (k)$. \square

Thanks to the following result by Delsarte and Lions, Corollary 3.12 extends to linear differential operators of finite order whose coefficients -except the leading one- may be non-constant.

Lemma 3.13. (Delsarte and Lions [23]) *Let $T : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ be a differential operator of the form $T = D^N + a_{N-1}(z)D^{N-1} + \cdots + a_0(z)I$, where $N \geq 1$ and $a_j \in H(\mathbb{C})$ for $1 \leq j \leq N$. Then there exists an onto isomorphism $U : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ so that $UT = D^N U$.*

Corollary 3.14. *For each $N \geq 1$ and $a_0, \dots, a_{N-1} \in H(\mathbb{C})$, the differential operator*

$$T = D^N + a_{N-1}(z)D^{N-1} + \cdots + a_0(z)I : H(\mathbb{C}) \rightarrow H(\mathbb{C})$$

has an \mathcal{U} -frequently hypercyclic subspace.

4. EXISTENCE OF COMMON HYPERCYCLIC SUBSPACES

4.1. Criterion (M_k) for common hypercyclic subspaces. In 2005, Bayart [4] generalized Criterion M_0 to the existence of common hypercyclic subspaces by using the Common Hypercyclicity Criterion of Costakis and Sambarino [22] and by using the approach introduced by Chan [20] to constructing hypercyclic subspaces via left-multiplication operators. With the same approach, Grosse-Erdmann and Peris showed that such Criterion M_0 for common hypercyclic subspaces remains true with their own version of the Common Hypercyclicity Criterion [28]. We note that a family of operators $\{T_\lambda\}_{\lambda \in \Lambda}$ satisfying the Common Hypercyclicity Criterion of Costakis and Sambarino [22] also satisfies the Common Hypercyclicity Criterion given in [28] and in particular it satisfies (CHC) given in Definition 1.7. We give next a constructive proof of Criterion M_0 for common hypercyclic subspaces for families of operators satisfying (CHC) .

Theorem 4.1 (Criterion M_0 for Common Hypercyclic Subspaces). *Let X be a Fréchet space with a continuous norm, let Y be a separable Fréchet space and let $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ be a family of sequences of operators in $L(X, Y)$ satisfying (CHC) and for which there exists an infinite-dimensional closed subspace M_0 such that for any $(\lambda, x) \in \Lambda \times M_0$,*

$$T_{n,\lambda}x \xrightarrow{n \rightarrow \infty} 0.$$

Then $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ has a common hypercyclic subspace.

Proof. Let $(p_j)_{j \geq 1}$ be an increasing sequence of norms inducing the topology of X , let $(q_j)_{j \geq 1}$ be an increasing sequence of seminorms inducing the topology of Y and let $K = [a, b] \subset \Lambda$. We denote by $X_{K,0}$, $Y_{K,0}$ and $S_{K,n,\lambda}$ the dense subsets and maps given by (CHC) for K . We first show that for any $\varepsilon > 0$, any $j \geq 1$, any $N_0 \geq 0$, and any $y \in Y_{K,0}$, there exist $N_1 \geq N_0$ and $x \in X$ with $p_j(x) < \varepsilon$ so that for any $\lambda \in [a, b]$ there exists $k \in [N_0, N_1]$ satisfying

$$(4.1) \quad q_j(T_{k,\lambda}x - y) < \varepsilon.$$

Let $\varepsilon > 0$, $j \geq 1$, $N_0 \geq 0$, and $y \in Y_{K,0}$ be given. We consider $C \geq N_0$ such that

(1) for any finite set $F \subset [C, \infty[$, any $l \geq 0$, any $\lambda \geq \mu_0 \geq \dots \geq \mu_l$ in K ,

$$q_j \left(\sum_{k \in F \cap [0, l]} T_{l,\lambda} S_{K,l-k,\mu_k} y \right) < \varepsilon;$$

(2) for any finite set $F \subset [C, \infty[$, any $l \geq 0$, any $\lambda \leq \mu_0 \leq \mu_1 \leq \dots$ in K ,

$$q_j \left(\sum_{k \in F} T_{l,\lambda} S_{K,l+k,\mu_k} y \right) < \varepsilon;$$

(3) for any finite set $F \subset [C, \infty[$, any $\mu_0 \leq \mu_1 \leq \dots$ in K .

$$p_j \left(\sum_{k \in F} S_{K,l,\mu_k} y \right) < \varepsilon.$$

Let $(\delta_l)_{l \geq 0}$ be a sequence of positive real numbers such that $\sum_{l=0}^{\infty} \delta_l = \infty$ and for any $\alpha, \lambda \in K$, any $l \geq 0$,

$$0 \leq \alpha - \lambda \leq \delta_l \quad \Rightarrow \quad q_j(T_{l,\lambda} S_{l,\alpha} y - y) < \varepsilon;$$

Since $\sum_{l=0}^{\infty} \delta_l = \infty$, there exists an increasing sequence $(k_l)_{l \geq 1}$ such that $\max\{C, N_0\} \leq k_1$, for any $l \geq 1$, $k_{l+1} - k_l \geq C$ and $\sum_{l=1}^{\infty} \delta_{k_l} = \infty$. Such a sequence exists because $(\{NC + k : N \geq 1\})_{k=0, \dots, N-1}$ forms a partition of $[C, \infty[$ and $\sum_{l=C}^{\infty} \delta_l = \infty$.

We select $L \geq 1$ the smallest integer such that

$$\sum_{l=1}^L \delta_{k_l} \geq b - a.$$

Let $\lambda_0 := a$ and $\lambda_l := \lambda_{l-1} + \delta_{k_l}$ for any $1 \leq l \leq L$. We deduce that $a = \lambda_0 < \lambda_1 < \dots < \lambda_{L-1} \leq b \leq \lambda_L$. We then consider

$$x = \sum_{l=0}^{L-1} S_{K,k_{l+1},\lambda_l} y \quad \text{and} \quad N_1 = k_L$$

and we show that x and N_1 satisfy the desired properties. We first remark that x satisfies

$$p_j(x) = p_j \left(\sum_{l=0}^{L-1} S_{K,k_{l+1},\lambda_l} y \right) < \varepsilon.$$

On the other hand, we see that for any $\lambda \in [a, b]$, there exists $0 \leq s \leq L-1$ such that $\lambda \in [\lambda_s, \lambda_{s+1}]$ and thus

$$\begin{aligned} q_j(T_{k_{s+1}, \lambda} x - y) &\leq q_j(T_{k_{s+1}, \lambda} S_{K, k_{s+1}, \lambda_s} y - y) \\ &\quad + q_j\left(\sum_{0 \leq l < s} T_{k_{s+1}, \lambda} S_{K, k_{l+1}, \lambda_l} y\right) + q_j\left(\sum_{s < l \leq L-1} T_{k_{s+1}, \lambda} S_{K, k_{l+1}, \lambda_l} y\right) \\ &< q_j(T_{k_{s+1}, \lambda} S_{K, k_{s+1}, \lambda_s} y - y) + 2\varepsilon \\ &\leq 3\varepsilon \quad \text{because } 0 \leq \lambda - \lambda_s \leq \lambda_{s+1} - \lambda_s \leq \delta_{k_{s+1}}. \end{aligned}$$

Since for any $0 \leq s \leq L-1$, $k_{s+1} \in [N_0, N_1]$, we conclude that (4.1) is satisfied.

Let M_0 be an infinite-dimensional closed subspace such that for any $\lambda \in \Lambda$, any $x \in M_0$,

$$T_{k, \lambda} x \xrightarrow[k \rightarrow \infty]{} 0.$$

We consider a chain $(K_n)_{n \geq 1}$ of Λ such that each K_n is a compact subinterval, and a basic sequence $(u_n)_{n \geq 1}$ in M_0 such that for every $n \geq 1$, we have $p_1(u_n) = 1$ and the sequence $(u_k)_{k \geq n}$ is basic in (X, p_n) with basic constant less than 2. We remark that, since each K_n is compact and $T_{l, \lambda}(x)$ depends continuously on λ , for any $n, k, j \geq 1$, there exist a positive number $C_{n, k, j}$ and a positive integer $m^0(n, k, j)$ such that for any $x \in X$,

$$(4.2) \quad \sup_{\lambda \in K_n} p_j(T_{k, \lambda} x) \leq C_{n, k, j} p_{m^0(n, k, j)}(x). \quad (\text{Banach-Steinhaus})$$

In particular, if \tilde{X} is a dense subset in X , we deduce from the previous reasoning that for any $\varepsilon > 0$, any $j, n \geq 1$, any $N_0 \geq 1$, any $y \in Y_{K_n, 0}$, there exists $x \in \tilde{X}$ and $N_1 \geq N_0$ such that $p_j(x) < \varepsilon$ and such that for any $\lambda \in K_n$, there exists $k \in [N_0, N_1]$ such that

$$(4.3) \quad q_j(T_{k, \lambda} x - y) < \varepsilon.$$

Let $(y_k)_{k \geq 1}$ be a dense sequence in X_0 and \prec the order on $\mathbb{N} \times \mathbb{N}$ defined by $(i, j) \prec (i', j')$ if $i + j < i' + j'$ or if $i + j = i' + j'$ and $i < i'$. We construct a family $(z_{i, j})_{i, j \geq 1} \subset X$ and two families $(n_{i, j}^0)_{i, j \geq 1}, (n_{i, j}^1)_{i, j \geq 1} \subset \mathbb{N}$ such that for any $i, j \geq 1$, $n_{i, j}^0 \leq n_{i, j}^1$ and for any $i \geq 1$, $(n_{i, j}^0)_{j \geq 1}, (n_{i, j}^1)_{j \geq 1}$ are increasing. If $z_{i', j'}$, $n_{i', j'}^0$ and $n_{i', j'}^1$ are already constructed for every $(i', j') \prec (i, j)$, then we choose $z_{i, j} \in X$ and $n_{i, j}^0, n_{i, j}^1$ with $n_{i, j}^0 > \max\{n_{i', j'}^1 : (i', j') \prec (i, j)\}$ such that

- we have,

$$(4.4) \quad \sum_{j' \leq j} z_{i, j'} \in X_{K_{i+j+1}, 0},$$

- for any $n \geq n_{i, j}^0$, any $i' \geq 1$ we have

$$(4.5) \quad \sup_{\lambda \in K_{i+j}} q_{i+j}\left(\sum_{j': (i', j') \prec (i, j)} T_{n, \lambda} z_{i', j'}\right) < \frac{1}{2^{i'+j}},$$

- we have

$$(4.6) \quad p_{i+j}(z_{i, j}) < \frac{1}{2^{i+j+2}},$$

- for any $(i', j') \prec (i, j)$, any $\lambda \in K_{i+j}$, any $n \in [n_{i', j'}^0, n_{i', j'}^1]$, we have

$$(4.7) \quad q_{i+j}(T_{n, \lambda} z_{i, j}) < \frac{1}{2^{i+j+j'}},$$

- for any $\lambda \in K_{i+j}$, there exists $n \in [n_{i,j}^0, n_{i,j}^1]$, such that we have

$$(4.8) \quad q_{i+j}(T_{n,\lambda} z_{i,j} - y_j) < \frac{1}{2^j}.$$

Satisfying (4.5) is possible by choosing $n_{i,j}^0$ sufficiently big because for any $i' \geq 1$, if $i' < i$,

$$\sum_{j': (i', j') \prec (i, j)} z_{i', j'} = \sum_{j' \leq i+j-i'} z_{i', j'} \in X_{K_{i+j+1}, 0}$$

and if $i' \geq i$,

$$\sum_{j': (i', j') \prec (i, j)} z_{i', j'} = \sum_{j' \leq i+j-i'-1} z_{i', j'} \in X_{K_{i+j}, 0}.$$

Satisfying (4.6) and (4.7) is possible by choosing $z_{i,j}$ sufficiently close to 0 thanks to (4.2), and we can choose $z_{i,j}$ and $n_{i,j}^1$ such that (4.4) and (4.8) are satisfied thanks to (4.3).

We define, for any $i \geq 1$,

$$z_i := u_i + \sum_{j=1}^{\infty} z_{i,j}.$$

By (4.6), these series are convergent and we deduce that the sequence $(z_i)_{i \geq 1}$ is a basic sequence equivalent to $(u_i)_{i \geq 1}$ in X . Let M be the closed linear span of (z_i) and $z \in M \setminus \{0\}$. We need to show that z is hypercyclic for each sequence $(T_{n,\lambda})$. Since (z_i) is a basic sequence, we can write $z = \sum_{i=1}^{\infty} \alpha_i z_i$ for some scalar sequence (α_i) , and re-scaling z if necessary we can further assume that $\alpha_k = 1$ for some k . By the equivalence between the basic sequences (z_i) and (u_i) , we deduce that $\sum_{i=1}^{\infty} \alpha_i u_i$ also converges and that there exists $K > 0$ such that for any $i \geq 1$ we have $|\alpha_i| \leq K$.

Let $l \geq 1$, $\lambda \in \Lambda$ and $r \geq l$ be given. If $\lambda \in K_r$ we deduce by (4.5), (4.7) and (4.8) that there exists $n \in [n_{k,r}^0, n_{k,r}^1]$ such that

$$\begin{aligned} q_l(T_{n,\lambda} z - y_r) &\leq q_l\left(\sum_{(i,j) \prec (k,r)} \alpha_i T_{n,\lambda} z_{i,j}\right) + q_l\left(\sum_{(i,j) \succ (k,r)} \alpha_i T_{n,\lambda} z_{i,j}\right) \\ &\quad + q_l(T_{n,\lambda} z_{k,r} - y_r) + q_l\left(T_{n,\lambda}\left(\sum_{i \geq 1} \alpha_i u_i\right)\right) \\ &\leq \sum_{i=1}^{\infty} K \left(q_{k+r}\left(\sum_{j: (i,j) \prec (k,r)} T_{n,\lambda} z_{i,j}\right) + \sum_{j: (i,j) \succ (k,r)} q_{i+j}(T_{n,\lambda} z_{i,j}) \right) \\ &\quad + q_{k+r}(T_{n,\lambda} z_{k,r} - y_r) + q_l\left(T_{n,\lambda}\left(\sum_{i \geq 1} \alpha_i u_i\right)\right) \\ &\leq \sum_{i=1}^{\infty} K \left(\frac{1}{2^{i+r}} + \sum_{j=1}^{\infty} \frac{1}{2^{i+j+r}} \right) + \frac{1}{2^r} + q_l\left(T_{n,\lambda}\left(\sum_{i \geq 1} \alpha_i u_i\right)\right) \\ &\leq \frac{2K+1}{2^r} + q_l\left(T_{n,\lambda}\left(\sum_{i \geq 1} \alpha_i u_i\right)\right). \end{aligned}$$

The last quantity can be made arbitrarily small as long as r is sufficiently large, since $\sum_{i \geq 1} \alpha_i u_i \in M_0$. So z is hypercyclic for $(T_{n,\lambda})_{n \geq 1}$. \square

Theorem 4.2 (Criterion (M_k) for Common Hypercyclic Subspaces). *Let X be a Fréchet space with a continuous norm, let Y be a separable Fréchet space and let $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ a family of sequences in $L(X, Y)$ satisfying (CHC) . Suppose there exist a chain (Λ_n) of Λ and a non-increasing sequence (M_k) of infinite-dimensional closed subspaces of X so that for each $n \in \mathbb{N}$,*

$$\{(T_{k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda_n} \text{ is uniformly equicontinuous along } (M_k).$$

Then $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ has a common hypercyclic subspace.

For the proof of Theorem 3.4 we used a map ϕ to guarantee that the subsequence $\{n_k : k \in I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]\}$ given by Corollary 2.6 kept the positiveness of the upper density of (n_k) . In the proof of Proposition 4.3 below we use the fact that given a divergent series $\sum_{k \geq 1} \delta_k$ of positive real numbers there exists a map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ so that for each increasing sequence of integers $(k_s)_{s \geq 1}$ in \mathbb{N} the set $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$ satisfies

$$\sum_{j \in I} \delta_j = \infty.$$

We prove Theorem 4.2 after the proof of Proposition 4.3.

Proposition 4.3. *Let X be a Fréchet space supporting a continuous norm, let Y be a separable Fréchet space and let $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ be a family of sequences in $L(X, Y)$ satisfying the (CHC) . Then there exists a map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that for any increasing sequence (k_s) in \mathbb{N} , if $(m_k) = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$, then $\{(T_{m_k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda}$ satisfies (CHC) .*

Proof. Let $(q_j)_{j \geq 1}$ be an increasing sequence of seminorms inducing the topology of Y . Let $(K_n)_{n \geq 1}$ be a chain of Λ such that each K_n is a compact subinterval. We denote by $X_0^{(n)}$, $Y_0^{(n)}$ and $S_{k,\lambda}^{(n)}$ the sets and maps given by (CHC) for the compact set K_n . We remark that without loss of generality we can assume that for each $n \geq 1$, $Y_0^{(n)}$ is countable. Moreover, for any $N \geq 1$, $j \geq 1$, $n \geq 1$, and $y_0 \in Y_0^{(n)}$ we denote by $(\delta_{y_0, N, j, k}^{(n)})_k$ a sequence of positive real numbers such that $\sum_{k=0}^{\infty} \delta_{y_0, N, j, k}^{(n)} = \infty$ and for each $k \geq 0$ and $\lambda, \mu \in K_N$ we have

$$0 \leq \mu - \lambda < \delta_{y_0, N, j, k} \Rightarrow q_j(T_{k,\lambda} S_{k,\mu}^{(n)} y_0 - y_0) < \frac{1}{N}.$$

We then choose ϕ such that for any $N \geq 1$, $j \geq 1$, $n \geq 1$, any $y_0 \in Y_0^{(n)}$, if $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$, then

$$\sum_{k \in I} \delta_{y_0, N, j, k}^{(n)} = \infty.$$

Such a map exists because, if $Y_0^{(n)} = (y_{n,l})_{l \geq 1}$, then it suffices to let $\phi(k)$ such that for any $n, j, l, N \leq k$,

$$\sum_{i=k}^{k+\phi(k)} \delta_{y_{n,l}, N, j, i}^{(n)} \geq 1.$$

Let (k_s) be an increasing sequence and $(m_k) = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$. We show that $\{(T_{m_k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda}$ satisfies (CHC) . We first remark that for any compact subset K in Λ , there exists $n \geq 1$ such that $K_n \supset K$. It thus suffices to prove that each

condition of (CHC) is satisfied for the compact subintervals K_n . Let $x_0 \in X_0^{(n)}$, $y_0 \in Y_0^{(n)}$ and q a continuous seminorm on Y .

- (1) Assume that for any finite set $F \subset [C, \infty[$, any $l \geq 0$, any $\lambda \geq \mu_0 \geq \dots \geq \mu_l$ in K_n ,

$$q\left(\sum_{k \in F \cap [0, l]} T_{l, \lambda} S_{l-k, \mu_k}^{(n)} y_0\right) < \varepsilon.$$

We remark that for any finite set $F \subset [C, \infty[$, any $l \geq C$, any $\lambda \geq \mu_0 \geq \dots \geq \mu_l$ in K_n , we have

$$q\left(\sum_{k \in F \cap [0, l-1]} T_{m_l, \lambda} S_{m_l-k, \mu_k}^{(n)} y_0\right) = q\left(\sum_{k' \in F' \cap [0, m_l]} T_{m_l, \lambda} S_{m_l-k', \nu_{k'}}^{(n)} y_0\right),$$

where $F' = \{k' \geq 0 : k' = m_l - m_{l-k}, k \in F \cap [0, l-1]\}$ and for any $k' \in F'$, if $k' = m_l - m_{l-k}$, then $\nu_{k'} = \mu_k$. In particular, we have $F' \subset [m_l - m_{l-C}, \infty[\subset [C, \infty[$ and thus we deduce that

$$q\left(\sum_{k \in F \cap [0, l-1]} T_{m_l, \lambda} S_{m_l-k, \mu_k}^{(n)} y_0\right) = q\left(\sum_{k' \in F' \cap [0, m_l]} T_{m_l, \lambda} S_{m_l-k', \nu_{k'}}^{(n)} y_0\right) < \varepsilon.$$

- (2) Assume that for any finite set $F \subset [C, \infty[$, any $l \geq 0$, any $\lambda \leq \mu_0 \leq \mu_1 \leq \dots$ in K_n ,

$$q\left(\sum_{k \in F} T_{l, \lambda} S_{l+k, \mu_k}^{(n)} y_0\right) < \varepsilon.$$

We remark that for any finite set $F \subset [C, \infty[$, any $l \geq 1$, and any $\lambda \leq \mu_0 \leq \mu_1 \leq \dots$ in K_n , we have

$$q\left(\sum_{k \in F} T_{m_l, \lambda} S_{m_l+k, \mu_k}^{(n)} y_0\right) = q\left(\sum_{k' \in F'} T_{m_l, \lambda} S_{m_l+k', \nu_{k'}}^{(n)} y_0\right),$$

where $F' = \{k' \geq 0 : k' = m_{l+k} - m_l, k \in F\}$ and for any $k' \in F'$, if $k' = m_{l+k} - m_l$, then $\nu_{k'} = \mu_k$. In particular, we have $F' \subset [m_{l+C} - m_l, \infty[\subset [C, \infty[$ and thus we deduce that

$$q\left(\sum_{k \in F} T_{m_l, \lambda} S_{m_l+k, \mu_k}^{(n)} y_0\right) = q\left(\sum_{k' \in F'} T_{m_l, \lambda} S_{m_l+k', \nu_{k'}}^{(n)} y_0\right) < \varepsilon.$$

- (3) Let $\varepsilon > 0$, $j \geq 1$ and $n \geq 1$. We consider $N \geq n$ such that $\frac{1}{N} < \varepsilon$. We know by choice of ϕ that $\sum_{k=1}^{\infty} \delta_{y_0, N, j, m_k} = \infty$, and we have, by definition of δ_{y_0, N, j, m_k} , that for each $k \geq 1$ and $\lambda, \mu \in K_N$,

$$0 \leq \mu - \lambda < \delta_{y_0, N, j, m_k} \Rightarrow q_j(T_{m_k, \lambda} S_{m_k, \mu}^{(n)} y_0 - y_0) < \varepsilon.$$

Since $K_N \supset K_n$, we have the desired result.

Conditions (4) and (5) are clear. \square

Proof of Theorem 4.2. Let (K_n) be a chain of Λ such that each K_n is a compact subinterval. We remark that the assumptions of Theorem 2.3 are satisfied if we consider $(\Lambda_n^0) = (\Lambda_n^1) = (K_n)$ and $(\Lambda_n^2) = (\Lambda_n)$. Indeed, we deduce that (Λ_n^0) satisfies the required properties by using the Banach-Steinhaus theorem and the fact that $T_{n, \lambda}$ depends continuously on λ .

On the other hand, as $\{(T_{n, \lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$ satisfies (CHC) , we deduce from Condition (4) in Definition 1.7 that for any subinterval $[a, b] \subset \Lambda$, there exists a dense

subset X_0 such that for any $x \in X_0$, $T_{k,\lambda}x \xrightarrow[k \rightarrow \infty]{} 0$ uniformly on $\lambda \in [a, b]$. The chain (Λ_n^1) thus satisfies the required properties.

We conclude by Proposition 4.3 and Theorem 2.3 that there exist an increasing sequence of positive integers (m_k) and an infinite-dimensional closed subspace M_0 of X such that $(T_{m_k,\lambda})_{k \geq 1}$ satisfies *(CHC)* and such that for each $x \in M_0$ and $\lambda \in \Lambda$,

$$T_{m_k,\lambda}x \xrightarrow[k \rightarrow \infty]{} 0.$$

We obtain the desired result by applying Theorem 4.1. \square

4.2. Applications of Criterion (M_k) for common hypercyclic subspaces.

In general, Criterion (M_k) is much easier to use than Criterion (M_0) because in a lot of cases, when Criterion (M_k) is satisfied, it is satisfied along the whole sequence (k) . We illustrate the use of Criterion (M_k) for common hypercyclic subspaces on families of weighted shifts on Köthe sequence spaces.

Definition 4.4. Let $A = (a_{j,k})_{j \geq 1, k \geq 0}$ be a matrix such that for any $j \geq 1$ and $k \geq 0$, we have $a_{j,k} > 0$ and $a_{j,k} \leq a_{j+1,k}$. The (real or complex) *Köthe sequence space* $\lambda^p(A)$ is defined as

$$\lambda^p(A) := \left\{ (x_k)_{k \geq 0} \in \omega : p_j((x_k)_k) = \left(\sum_{k=0}^{\infty} |x_k a_{j,k}|^p \right)^{\frac{1}{p}} < \infty, j \geq 1 \right\},$$

endowed with the sequence of norms (p_j) .

Let Λ be an open interval of \mathbb{R} and w_λ be a sequence of non-zero scalars. The weighted shift B_{w_λ} is defined as $B_{w_\lambda}e_n = w_{\lambda,n}e_{n-1}$, where $e_{-1} = 0$ and $(e_n)_{n \geq 0}$ is the canonical basis. We provide in Proposition 4.6 a sufficient condition for a family of shifts $\{B_{w,\lambda}\}_{\lambda \in \Lambda}$ satisfying the *(CHC)* on $\lambda^p(A)$ to support a common hypercyclic subspace. We first note the following.

Remark 4.5. Let $(p_j)_{j \geq 1}$ and $(q_j)_{j \geq 1}$ be increasing sequences of seminorms inducing the topologies of X , and Y , respectively. If $T_{n,\lambda}$ depends continuously on λ and K is a compact subset of Λ , then $\{(T_{n,\lambda})_{n \geq 1}\}_{\lambda \in K}$ is uniformly equicontinuous along (M_k) if and only if for any $j \geq 1$, there exists $C_j > 0$ and $k(j), m(j) \geq 1$ such that for any $k \geq k(j)$, any $\lambda \in K$ and any $x \in M_k$,

$$q_j(T_{k,\lambda}x) \leq C_j p_{m(j)}(x).$$

Proposition 4.6. Let $\lambda^p(A)$ be a Köthe sequence space, Λ an open interval of \mathbb{R} and let $\{B_{w_\lambda}\}_{\lambda \in \Lambda}$ be a family of weighted shifts on $\lambda^p(A)$ satisfying *(CHC)*. If there exist a chain of compact sets $(K_n)_{n \geq 1}$ of Λ , and for each $n \geq 1$ a sequence $(C_{n,j})_j$ of positive scalars and a sequence $(m(n,j))_j$ of integers such that for any $l \geq 1$ and any $N \geq 0$,

$$\inf_{k \geq N} \max_{1 \leq n, j, m \leq l} \sup_{\lambda \in K_n} \frac{p_j(B_{w_\lambda}^m e_k)}{C_{n,j} p_{m(n,j)}(e_k)} \leq 1,$$

then $\{B_{w_\lambda}\}_{\lambda \in \Lambda}$ has a common hypercyclic subspace.

Proof. Let (K_n) , $(C_{n,j})$ and $(m(n,j))$ be sequences satisfying the above assumptions. For any $l \geq 1$, we consider e_{n_l} such that $n_l > n_{l-1}$ and for any $n, j, m \leq l$,

$$\sup_{\lambda \in K_n} \frac{p_j(B_{w_\lambda}^m e_{n_l})}{p_{m(n,j)}(e_{n_l})} \leq 2C_{n,j},$$

Let $M_k = \overline{\text{span}}\{e_{n_l} : l \geq k\}$. We deduce that for any $n, j \geq 1$, any $\lambda \in K_n$, any $k \geq \max\{n, j\}$ and any $x \in M_k$ we have

$$\begin{aligned} p_j(B_{w_\lambda}^k x)^p &= p_j\left(B_{w_\lambda}^k \left(\sum_{l=k}^{\infty} x_{n_l} e_{n_l}\right)\right)^p \\ &= \sum_{l=k}^{\infty} |x_{n_l}|^p p_j(B_{w_\lambda}^k e_{n_l})^p \\ &\leq (2C_{n,j})^p \sum_{l=k}^{\infty} |x_{n_l}|^p p_{m(n,j)}(e_{n_l})^p \\ &\leq (2C_{n,j})^p p_{m(n,j)}(x)^p. \end{aligned}$$

Since each K_n is compact and B_{w_λ} depends continuously on λ , we conclude by Remark 4.5 that each $\{B_{w_\lambda}\}_{\lambda \in K_r}$ is uniformly equicontinuous along (M_k) . The conclusion now follows by Theorem 4.2. \square

Corollary 4.7. *Let $\lambda^p(A)$ be a Köthe sequence space. Let Λ be an open interval of \mathbb{R} and let $\{B_{w_\lambda}\}_{\lambda \in \Lambda}$ be a family of weighted shifts on $\lambda^p(A)$ satisfying (CHC). Suppose that for each compact subset K of Λ there exist a sequence $(C_{K,j})$ of positive scalars and a sequence $(m(K,j))$ of positive integers such that for any $j \geq 1$ and $n \geq 1$ we have*

$$\limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \frac{p_j(B_{w_\lambda}^n e_k)}{C_{K,j} p_{m(K,j)}(e_k)} \leq 1.$$

Then $\{B_{w_\lambda}\}_{\lambda \in \Lambda}$ has a common hypercyclic subspace.

Corollary 4.7 gives an answer to a question of Costakis and Sambarino [22, Section 8]:

Corollary 4.8. *For each $\lambda > 1$, let B_{w_λ} be the backward shift on ℓ^p ($1 \leq p < \infty$) with weight sequence $w_\lambda = (1 + \frac{\lambda}{k})_{k \geq 1}$. Then $\{B_{w_\lambda}\}_{\lambda > 1}$ has a common hypercyclic subspace.*

Proof. Costakis and Sambarino [22] have shown that the family $\{B_{w_\lambda}\}_{\lambda > 1}$ satisfies (CHC). But for each $K = [a, b] \subset (0, \infty)$ and $n \geq 1$ we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \|B_{w_\lambda}^n e_k\| &= \limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \left(\prod_{\nu=0}^{n-1} |w_{\lambda, k-\nu}| \right) \\ &= \limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \left(\prod_{\nu=0}^{n-1} \left(1 + \frac{\lambda}{k-\nu} \right) \right) \\ &\leq \limsup_{k \rightarrow \infty} \left(1 + \frac{b}{k-n+1} \right)^n = 1. \end{aligned}$$

That is, the assumptions of Corollary 4.7 are satisfied. \square

Bayart and Ruzsa [9] showed that a shift B_w with weight sequence $w = (w_n)$ is frequently hypercyclic on ℓ^p if and only if it is \mathcal{U} -frequently hypercyclic and if and only if $(\frac{1}{\prod_{1 \leq j \leq n} w_j})_{n \geq 1} \in \ell^p$. So each shift B_{w_λ} from Corollary 4.8 is frequently hypercyclic on ℓ^p for any $1 \leq p < \infty$. Moreover, by Corollary 3.9, each B_{w_λ} has a \mathcal{U} -frequently hypercyclic subspace on ℓ^p . This motivates the following.

Problem 2. For each $\lambda \in \mathbb{C}$, let $\{B_{w_\lambda}\}_{\lambda>1}$ be the shift operator on ℓ^p with weight sequence $w_\lambda = (1 + \frac{\lambda}{n})$. Does $\{B_{w_\lambda}\}_{\lambda>1}$ have a common \mathcal{U} -frequently hypercyclic subspace?

We note that two operators having \mathcal{U} -frequently hypercyclic subspaces may fail to have a common hypercyclic subspace.

Example 4.9. Let B_w be the backward shift operator on ℓ^2 of weight sequence $w = (\frac{n+1}{n})$. Then each of the operators $T_1 = B_w \oplus 2B$ and $T_2 = 2B \oplus B_w$ on $X = \ell^2 \oplus \ell^2$ has a \mathcal{U} -fhc subspace, but $\{T_1, T_2\}$ has no common hypercyclic subspace. The latter follows from an argument used in [1, Example 2.1]: For $j = 1, 2$, let $P_j : \ell^2 \oplus \ell^2 \rightarrow \ell^2$, $P_j(x_1, x_2) = x_j$ be the j -th orthogonal projection. If $M \subset \ell^2 \oplus \ell^2$ is a hypercyclic subspace for both T_1 and T_2 , then $P_1(M) \cup P_2(M)$ consists (but zero) of common hypercyclic vectors for B_w and $2B$. But since M is closed and infinite-dimensional and X is a Hilbert space, one of $P_1(M)$, $P_2(M)$ must contain a closed and infinite-dimensional subspace, contradicting the fact that $2B$ has no hypercyclic subspaces. It remains to see that each of T_1 and T_2 has a \mathcal{U} -fhc subspace. Each of T_1, T_2 satisfies the FHC on $\ell^2 \oplus \ell^2$, since each of their summands B_w and $2B$ satisfies the FHC on ℓ^2 . Also, B_w has a hypercyclic subspace on ℓ^2 , and there exists a non-increasing sequence (M_k) of closed and infinite-dimensional subspaces of ℓ^2 for which $\sup_{k \geq 1} \|B_w|_{M_k}\| < \infty$. Thus (T_1^k) and (T_2^k) are equicontinuous along $(M_k \oplus \{0\})$ and $(\{0\} \oplus M_k)$, respectively, and the conclusion follows from Theorem 3.7

In 2010, Shkarin [40] showed that the derivative operator D on $H(\mathbb{C})$ has a hypercyclic subspace. By considering $H(\mathbb{C})$ as the Köthe sequence space $\Lambda^1(A)$ for $a_{j,k} = j^k$, we can now prove the existence of a common hypercyclic subspace for the family of operators $\{\lambda D\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ on $H(\mathbb{C})$.

Corollary 4.10. *Let D be the derivative operator on $H(\mathbb{C})$. Then the family $\{\lambda D\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ has a common hypercyclic subspace.*

Proof. We know that the family $\{\lambda D\}_{\lambda>0}$ satisfies (CHC) [22]. Moreover, for any $K = [a, b] \subset]0, \infty[$ and any $n \geq 1$ we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \frac{p_j(\lambda^n D^n e_k)}{p_{2j}(e_k)} &= \limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \frac{p_j(\lambda^n D^n e_{k+n})}{p_{2j}(e_{k+n})} \\ &= \limsup_{k \rightarrow \infty} \sup_{\lambda \in K} \frac{\lambda^n \prod_{\nu=1}^n (k + \nu) p_j(e_k)}{p_{2j}(e_{k+n})} \\ &\leq \limsup_{k \rightarrow \infty} \frac{b^n (k+n)^n j^k}{(2j)^{k+n}} = 0. \end{aligned}$$

Thus by Corollary 4.7 the family $\{\lambda D\}_{\lambda>0}$ has a common hypercyclic subspace. The conclusion now follows from the fact that for any complex scalar λ and operator T the operators λT and $|\lambda|T$ have the same set of hypercyclic vectors [30]. \square

Corollary 4.10 extends to operators $P(D)$ where P is a non-constant polynomial.

Proposition 4.11. *Let D be the derivative operator on $H(\mathbb{C})$ and let P be a non-constant polynomial. Then $\{\lambda P(D)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ has a common hypercyclic subspace.*

Proof. The family $\{(\lambda^k P(D)^k)_{k \geq 1}\}_{\lambda>0}$ satisfies (CHC) (see the proof of [21, Proposition 2.2] and Remark 1.9). On the other hand, we know [32] that there exists a

non-increasing sequence of infinite-dimensional closed subspaces (M_k) in $H(\mathbb{C})$ such that for any $j \geq 1$, there exist a positive number C_j and two integers $m(j), k(j)$ such that for any $k \geq k(j)$, any $x \in M_k$,

$$p_j(P(D)^k x) \leq C_j p_{m(j)}(x).$$

Without loss of generality, we can assume that for any $k \geq 1$, the subspace M_k is included in $\overline{\text{span}}\{e_n : n \geq k\}$. We recall that the sequence (p_j) of norms considered here is given by $p_j((x_k)_k) = \sum_{k=0}^{\infty} |x_k j^k|$. In particular, we deduce from the previous inclusion that for any $x \in M_k$, any $j \geq 1$, any $n \geq 1$,

$$p_j(x) \leq n^{-k} p_{nj}(x).$$

Let $n \geq 1$. If we consider $\Lambda_n^2 = [\frac{1}{n}, n]$, $C_{n,j} = C_j$, $m(n, j) = nm(j)$, $k(n, j) = k(j)$, then for any $\lambda \in \Lambda_n^2$, any $k \geq k(n, j)$, any $x \in M_k$, we have

$$p_j((\lambda P(D))^k x) \leq \lambda^k C_j p_{m(j)}(x) \leq \lambda^k C_j n^{-k} p_{nm(j)}(x) \leq C_j p_{m(n,j)}(x).$$

By Remark 4.5 and Theorem 4.2, the family $\{\lambda P(D)\}_{\lambda > 0}$ has a common hypercyclic subspace. So $\{\lambda P(D)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ has a common hypercyclic subspace, as for any operator T and any complex scalar λ the operators λT and $|\lambda|T$ have the same set of hypercyclic vectors [30]. \square

As with Corollary 3.14, Lemma 3.13 by Delsarte and Lions allows to extend Proposition 4.11 to any linear differential operators of finite order whose coefficients -other than the leading one- may be non-constant.

Corollary 4.12. *For each $N \geq 1$ and $a_0, \dots, a_{N-1} \in H(\mathbb{C})$, consider the differential operator $T = D^N + a_{N-1}(z)D^{N-1} + \dots + a_0(z)I : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. Then*

$$\{\lambda T\}_{\lambda > 0}$$

has a common hypercyclic subspace.

Recall that for any non-constant entire function ϕ of exponential type the family $\{\lambda \phi(D)\}_{\lambda > 0}$ has a common hypercyclic vector. This is due to Costakis and Mavroudis [21] for the case ϕ is a polynomial, to Bernal [11] when ϕ has exponential growth order no larger than $\frac{1}{2}$, and due to Shkarin [39] in the general case. Proposition 4.11 motivates the following.

Problem 3. Let ϕ be entire, transcendental, and of exponential type. Does the family $\{\lambda \phi(D)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ have a common hypercyclic subspace?

4.3. Multiples of an operator on a complex Banach space. A characterization by González, León and Montes [26] asserts that an operator T on a complex Banach space satisfying the Hypercyclicity Criterion has a hypercyclic subspace if and only if its essential spectrum intersects the closed unit disc. We recall that if the essential spectrum of T does not intersect the closed unit disk then for any infinite-dimensional closed subspace M , there exists $x \in M$ such that $\|T^n x\| \rightarrow \infty$. We denote by $\sigma_e(T)$ the essential spectrum of T , i.e. the set of complex scalars λ such that $\lambda Id - T$ is not a Fredholm operator, where an operator is said to be Fredholm if its range is closed and its kernel and cokernel are finite-dimensional.

While it is likely not possible to extend such spectral characterization to arbitrary families of operators (even if they consist of two operators) as evidenced in [14], we show that it indeed extends to certain families of the form $\{\lambda T\}_{\lambda \in \Lambda}$ thanks to Criterion (M_k) for common hypercyclic subspaces. Throughout this section, we let

X be a complex Banach space, $T \in L(X)$ and let $\mathcal{P} = (P_\lambda)_{\lambda \in \Lambda}$ be a family of polynomials. We denote by

$$r_{\mathcal{P}} := \inf\{r > 0 : \exists \lambda \in \Lambda, P_\lambda(B(0, r)^c) \cap \overline{B(0, 1)} = \emptyset\}.$$

We start by stating a sufficient condition for the non-existence of common hypercyclic subspaces for the family $\{P_\lambda(T)\}_{\lambda \in \Lambda}$ in terms of the essential spectrum of T .

Proposition 4.13. *If $\sigma_e(T) \cap \overline{B(0, r_{\mathcal{P}})} = \emptyset$ (with $\overline{B(0, 0)} = \{0\}$), then there exists $\lambda \in \Lambda$ such that each infinite-dimensional closed subspace M of X contains some x for which*

$$\|P_\lambda^n(T)x\| \xrightarrow{n \rightarrow \infty} \infty.$$

In particular, there exists $\lambda \in \Lambda$ such that $P_\lambda(T)$ has no hypercyclic subspace.

Proof. Since $\sigma_e(T)$ is compact and $\sigma_e(T) \cap \overline{B(0, r_{\mathcal{P}})} = \emptyset$, there exists $r > r_{\mathcal{P}}$ such that

$$\sigma_e(T) \cap B(0, r) = \emptyset.$$

By definition of $r_{\mathcal{P}}$, we know that there exists $\lambda \in \Lambda$ such that

$$P_\lambda(B(0, r)^c) \cap \overline{B(0, 1)} = \emptyset.$$

Therefore, since $\sigma_e(P_\lambda(T)) = P_\lambda(\sigma_e(T))$, we deduce that $\sigma_e(P_\lambda(T)) \subset P_\lambda(B(0, r)^c)$ and thus

$$\sigma_e(P_\lambda(T)) \cap \overline{B(0, 1)} = \emptyset,$$

i.e. the essential spectrum of $P_\lambda(T)$ does not intersect the closed unit disk. We conclude that for any infinite-dimensional closed subspace $M \subset X$, there exists $x \in M$ such that

$$\|P_\lambda^n(T)x\| \xrightarrow{n \rightarrow \infty} \infty.$$

□

Under certain conditions on the essential spectrum of T , we can construct a non-increasing sequence (M_n) of closed infinite-dimensional subspaces of X so that the assumptions of Criterion (M_k) for common hypercyclic subspaces are met.

Proposition 4.14. *Suppose $\sup_{\lambda \in \Lambda} |P_\lambda(\mu)| \leq 1$ and that $\text{Ran}(T - \mu Id)$ is dense in X , for some $\mu \in \sigma_e(T)$. Then there exists a non-increasing sequence of infinite-dimensional closed subspaces (M_n) in X and a chain (Λ_n) of Λ such that for any $n \geq 1$, any $\lambda \in \Lambda_n$, any $m \leq n$, we have*

$$\|P_\lambda^m(T)x\| \leq 2\|x\| \quad \text{for any } x \in M_n.$$

Proof. Since $\text{Ran}(T - \mu Id)$ is dense and $\mu \in \sigma_e(T)$, either $\dim \ker(T - \mu Id) = \infty$ or $\text{Ran}(T - \mu Id)$ is not closed. If $\dim \ker(T - \mu Id) = \infty$, then we consider $M_n = \ker(T - \mu Id)$. Therefore, for any $n \geq 1$, any $x \in M_n$, and any $\lambda \in \Lambda$ we have

$$\|P_\lambda^n(T)x\| = \|(P_\lambda(\mu))^n x\| \leq \|x\|.$$

We now suppose that $\dim \ker(T - \mu Id) < \infty$ and $\text{Ran}(T - \mu Id)$ is not closed. We can then show that there exists an infinite-dimensional closed subspace M in X and a compact operator K such that $T|_M = \mu Id + K$ (see [26]). Therefore, for any $n \geq 0$ we have $T|_M^n = \mu^n Id + K_n$ where K_n is a compact operator. Since K_n is compact, we know that for any $\varepsilon > 0$, there exists a closed subspace of finite-codimension $E_{n,\varepsilon}$ in X such that $\|K_n|_{E_{n,\varepsilon}}\| \leq \varepsilon$.

We consider

$$\Lambda_n = \{\lambda \in \Lambda : \deg P_\lambda \leq n \text{ and } P_\lambda(x) = \sum_{k=0}^n a_k x^k \text{ with } \sum_{k=0}^n |a_k| \leq n\}$$

and we let $M_0 := M$ and

$$M_n := M_{n-1} \cap \bigcap_{k \leq n^2} E_{k, \varepsilon_n} \quad \text{with} \quad \varepsilon_n = \frac{1}{n^n}.$$

We remark that for any $n \geq 1$, any $\lambda \in \Lambda_n$, any $m \geq 0$,

$$P_\lambda^m(T) = \sum_{k=0}^{mn} c_k T^k \quad \text{with} \quad \sum_{k=0}^{mn} |c_k| \leq n^m.$$

We deduce that for any $n \geq 1$, any $\lambda \in \Lambda_n$, any $m \leq n$, and any $x \in M_n$,

$$\begin{aligned} \|P_\lambda^m(T)x\| &= \left\| \sum_{k=0}^{mn} c_k T^k x \right\| = \left\| \sum_{k=0}^{mn} c_k (\mu^k x + K_k x) \right\| \\ &\leq |P_\lambda(\mu)|^m \cdot \|x\| + \sum_{k=0}^{mn} (|c_k| \cdot \|K_k|_{M_n}\|) \|x\| \leq \|x\| + n^n \varepsilon_n \|x\| \leq 2\|x\|. \end{aligned}$$

□

Thanks to Proposition 4.13 and Proposition 4.14, we can now generalize the characterization of González, León and Montes as follows:

Theorem 4.15. *Let X be a separable infinite-dimensional complex Banach space, $T \in L(X)$ and $\{P_\lambda\}_{\lambda \in \Lambda}$ a family of polynomials. If $\{P_\lambda(T)\}_{\lambda \in \Lambda}$ satisfies (CHC) and if for any $\mu \in \overline{B(0, r_{\mathcal{P}})}$, $\text{Ran}(T - \mu \text{Id})$ is dense and $\sup_{\lambda \in \Lambda} |P_\lambda(\mu)| \leq 1$, then the following assertions are equivalent:*

- (1) *for any $\lambda \in \Lambda$, $P_\lambda(T)$ has a hypercyclic subspace;*
- (2) *$\{P_\lambda(T)\}_{\lambda \in \Lambda}$ has a common hypercyclic subspace;*
- (3) *$\{P_\lambda(T)\}_{\lambda \in \Lambda}$ satisfies Criterion M_0 for common hypercyclic subspaces;*
- (4) *$\{P_\lambda(T)\}_{\lambda \in \Lambda}$ satisfies Criterion (M_k) for common hypercyclic subspaces;*
- (5) *the essential spectrum of T intersects $\overline{B(0, r_{\mathcal{P}})}$.*

Proof. $\neg(5) \Rightarrow \neg(1)$ follows from Proposition 4.13.

$(5) \Rightarrow (4)$. If the essential spectrum of T intersects $\overline{B(0, r_{\mathcal{P}})}$, then by assumption, there exists $\nu \in \sigma_e(T)$ such that $\text{Ran}(T - \nu \text{Id})$ is dense and $\sup_{\lambda \in \Lambda} |P_\lambda(\nu)| \leq 1$. Since $(P_\lambda(T))_{\lambda \in \Lambda}$ satisfies (CHC), we deduce from Proposition 4.14 that $(P_\lambda(T))_{\lambda \in \Lambda}$ satisfies the Criterion (M_k) for common hypercyclic subspaces.

$(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ follow from Theorem 4.2 and Theorem 4.1.

$(2) \Rightarrow (1)$ is immediate. □

In the case of scalar multiples of an operator, Theorem 4.15 gives us the following two interesting characterizations.

Corollary 4.16. *Let $T \in L(X)$, where X is a separable infinite-dimensional complex Banach space, and let $0 < a < b$. If $\{\lambda T\}_{\lambda \in]a, b[}$ satisfies (CHC), then the following assertions are equivalent:*

- (1) *the family $\{\lambda T\}_{\lambda \in]a, b[}$ has a common hypercyclic subspace;*
- (2) *for any $\lambda \in]a, b[$, the operator λT has a hypercyclic subspace;*
- (3) *the essential spectrum of T intersects $\overline{B(0, \frac{1}{b})}$.*

Proof. Let $\mathcal{P} = \{\lambda Id : \lambda \in]a, b[\}$. We remark that $r_{\mathcal{P}} = \frac{1}{b}$ and for any $\mu \in \overline{B(0, r_{\mathcal{P}})} = \overline{B(0, \frac{1}{b})}$, we have

$$\sup_{\lambda \in]a, b[} |\lambda \mu| = b|\mu| \leq 1.$$

Moreover, since $\{\lambda T\}_{\lambda \in]a, b[}$ satisfies (CHC), we know in particular that λT is hypercyclic for any $\lambda \in]a, b[$ and therefore that $\text{Ran}(T - \mu Id)$ is dense for any $\mu \in \mathbb{C}$ [19]. The conclusion follows by Theorem 4.15. \square

Corollary 4.17. *Let $T \in L(X)$, where X is a separable infinite-dimensional complex Banach space, and let $a > 0$. If $\{\lambda T\}_{\lambda \in]a, \infty[}$ satisfies (CHC), then the following assertions are equivalent:*

- (1) *the family $\{\lambda T\}_{\lambda \in]a, \infty[}$ has a common hypercyclic subspace;*
- (2) *for any $\lambda > a$, the operator λT has a hypercyclic subspace;*
- (3) $0 \in \sigma_e(T)$.

Proof. If $\mathcal{P} = \{\lambda Id : \lambda \in]a, \infty[\}$, we have $r_{\mathcal{P}} = 0$ and thus $\overline{B(0, r_{\mathcal{P}})} = \{0\}$. We conclude as previously by using Theorem 4.15. \square

We note that Corollary 4.17 applies to several interesting examples by Gallardo and Partington [24, Section 3] of families of scalar multiples of either adjoint multiplication operators on the Hardy space, adjoint multiplier operators on weighted ℓ^2 -spaces, adjoint convolution operators on weighted $L^2(0, \infty)$ spaces, or adjoint composition operators on the reduced Hardy space. This is because any family $\{\lambda T\}_{|\lambda| > a}$ satisfying the assumptions of [24, Theorem 2.1] must satisfy (CHC). We illustrate with the following.

Example 4.18. Suppose $\varphi \in H^\infty(\mathbb{D})$ is univalent and bounded below on \mathbb{T} but is not an outer function. Suppose that zero belongs to the boundary of $\varphi(\mathbb{D})$. Then

$$\{\lambda M_\varphi^*\}_{|\lambda| > a}$$

has a common hypercyclic subspace on $H^2(\mathbb{D})$, where $a = \|\frac{1}{\varphi}\|_{L^\infty_{\mathbb{T}}}$.

Proof. We know that the spectrum of M_φ^* is the closure of $\{\overline{\varphi(z)} : z \in \mathbb{D}\}$ and that its essential spectrum is $\sigma_e(M_\varphi^*) = \partial\{\overline{\varphi(z)} : z \in \mathbb{D}\}$ thanks to φ being univalent, see [25, Proposition 4.4(b)]. Hence the assumption $0 \in \partial\varphi(\mathbb{D})$ gives that $0 \in \sigma_e(M_\varphi^*)$, and the conclusion follows by Corollary 4.17. \square

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